Math 1A: Calculus	**Instructor: Alexander Paulin**

Quiz 7: §3.8-4.2

Your name: *Discussions 201, 203 // 2018-10-26*

Problem 1 (7 points, 1 point per entry)**.** In this problem, f is a function that is continuous on the interval $[2, 8]$, and whose critical numbers *occur only at integers*.

Using this information, fill out the blank (white) cells in the below table. Ignore the grayed out cells.

x	2	3	4	5	6	7	8
$f(x)$	-5		0	5		1	7
$f'(x)$		DNE	3		0		
critical pt.? (yes/no)		yes	no	yes	yes	yes	
local min/max or neither		neither	neither	local max	neither	local min	

Solution: Recall that a critical point is where the derivative is either 0 or does not exist, and local maxima/minima must occur at critical points.

The rightmost two cells are tricky. I was hoping people would draw some pictures and convince themselves that f had to have a local minimum at $x = 7$. Here's a precise argument: we know that f is continuous on $[5, 8]$ so the Extreme Value Theorem says that f attains an absolute minimum on $[5, 8]$. But $f(7)$ is smaller than both $f(5)$ and $f(8)$ so that means the absolute minimum is NOT at an endpoint. So there's a local minimum, and such a thing must occur at a critical point. Moreover, the problem stipulates that all critical numbers of f are integers. Since the table says f does *not* have a local minimum at $x = 6$, it must be at $x = 7$. □

Problem 2 (4 points)**.** Explain why there CANNOT be a function f with all of the following properties:

- f is differentiable on all of $\mathbb{R}$,
- $f(1) = 2$,
- $f(5) = 22$,
- $f'(x) < 4$ for all x.

Solution: Suppose that such a function f existed. Then by the Mean Value Theorem applied to the interval $[1, 5]$ (f is differentiable everywhere so also continuous everywhere) we conclude that there exists some c in $(1, 5)$ for which

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{20}{4} = 5.$$

But that contradicts the requirement that $f'(x) < 4$ for all x. Hence such a function f cannot exist. □

(Continued on back of sheet.)

Problem 3 (2 + 2 = 4 points)**.** Alice takes a water bottle from her room (21 degrees Celsius) and puts it in her fridge (5 degrees Celsius) at 7:00 A.M. At 7:10 A.M., the temperature of the water has dropped to 17 degrees Celsius.

(a) Express the temperature of the water bottle as a function of time $t \geq 0$ (with t measured in minutes after 7:00 A.M.).
(b) At what time is the water 14 degrees Celsius?

Setup. This "setup" part in violet is stuff I honestly wasn't expecting people to do on the quiz, but seeing as to how a lot of people tried to start from here, I figured I'd start from here too.

Newton's law of cooling says that our water temperature T obeys a differential equation of the form

$$\frac{dT}{dt} = k(T - T_s)$$

where T_s is the temperature of the surroundings—in this case the fridge, which is at 5 degrees. Let $y = T - T_s$, so that

$$\frac{dy}{dt} = \frac{d}{dt}(T - T_s) = \frac{dT}{dt} = k(T - T_s) = ky.$$

We know that the only solutions to this differential equation have the form Ae^{kt}. This means that

$$T = y + T_s = Ae^{kt} + T_s$$

for some constants A, k to be determined. The above derivation should be familiar to you from lecture, and **I wasn't expecting you to reproduce them on the quiz**. □

Solution:

(a) Basically, I was hoping that people would recognize the temperature would be given by an equation of the form

$$T(t) = Ae^{kt} + T_s.$$

Here $T_s = 5$. We know that $T(0) = 21$, i.e.

$$T(0) = Ae^0 + 5 = A + 5 = 21$$

so $A = 16$. Then plugging in $t = 10$ lets us solve for the last unknown constant, which is k:

$$\begin{aligned} 17 &= 16e^{10k} + 5 \\ \frac{3}{4} &= e^{10k} \\ \frac{1}{10}\ln\frac{3}{4} &= k. \end{aligned}$$

Hence

$$\boxed{T(t) = 16e^{\frac{t}{10}\ln\frac{3}{4}} + 5 = 16\left(\frac{3}{4}\right)^{t/10} + 5.}$$

(I don't care if you wrote it as the first thing as opposed to the second.)

(b) Solve $T(t) = 14$:

$$\begin{aligned} 16\left(\frac{3}{4}\right)^{t/10} + 5 &= 14 \\ \left(\frac{3}{4}\right)^{t/10} &= \frac{9}{16} \\ \frac{t}{10} &= 2 \end{aligned}$$

so $t = 20$, meaning that the water will be 14 degrees Celsius at $\boxed{\text{7:20 A.M.}}$ If you left your answer as 20 (or any expression that is equal to 20) that's fine too.

□